Three Tables for Bob

by Marc Sabat

2016

To distinguish differences of harmony, to compose music so that players and listeners may sense these variations naturally, in real time, the representation of harmonic space in musical notation must be precise yet simple to read. Shadings of intonation are most clearly conceived in relation to consonant, untempered intervals. Intervals derived from a common frame of reference may be represented by microtonal signs, which suggest potentially tuneable intervallic relationships.

Successive intervals, however, often accumulate an illegible proliferation of microtonal signs and the melodic direction of pitches becomes unclear. How may this paradox between exactness and practical realisation be bridged? How would it be possible to compose a harmonic microtonal music that moves freely, that is not forced by notational limitations to constantly retrace steps?

The current equal tempered system may be modified with cent deviations to more closely approximate microtonal shadings and bendings of intonation, but this approach does not reveal the manifold harmonic relationships clustered around a particular microtonal pitch. To see these relationships, and thereby discover possible notational substitutions and simplifications, why not begin by mapping the unfolding of harmonic space itself? Taking the simplest Pythagorean pitches as a common starting point, what enharmonic near-equivalences emerge naturally from successions of rational intervals?

Thinking about these questions, and how they might inspire the composition of ensemble music in extended Just Intonation, led me to sketch out the three tables presented in this article, dedicated to the remarkable microtonal theorist and musician Bob Gilmore.

0. Harmony: the interaction and intonation of tonal sounds

Different frequencies of vibration coexisting — in a sounding body, a resonator or in the air itself — produce interferences, which are perceived as audible patterns: characteristic beatings and combination tones. *Intonation* is the art of learning to recognise, distinguish, produce and shape composite rhythms and inner melodies within any combination of sounds: namely, to make *harmony*. To explore harmony in an experimental way is to investigate how these interactions of tone are sensed, how minute changes and shadings of intonation affect the perceptions of beating, fusion, spatialisation, colouration, orchestration, consonance and dissonance within sound-aggregates.

Tonal sounds are any sounds heard as *pitches*, and are generally composed of a spectrum of *harmonic partials*, whole number multiples of a *fundamental* vibrating frequency. In the special case of an electronically generated sinewave, only the fundamental is sounding. Our brain perceives tonal sounds by grouping harmonically related partials with similar morphologies to reconstruct layerings of individual *timbres*. For example, it is possible to distinguish a clarinet playing while a human voice sings a particular vowel, even on the same pitch.

Some tonal sounds, like those produced by a piano, include partials that deviate from an ideal series of harmonics. However, even so-called inharmonic mixtures, like some multiphonics or gong-like sounds, in which several fundamentals are perceived, may be analysed by the hearing process as a superposition of (possibly detuned) *intervals* definable as frequency ratios.

Tonal sounds may be sensed as being proximate when some of their respective partials vibrate at the same, or nearly the same, frequencies. If this is true of their perceived fundamentals, the *melodic distance* between the sounds is small, that is, there is nearly a *unison* or alignment of all partials. When a unison takes place between some of the higher partials, even though the sounds' fundamentals differ, another kind of proximity may be perceived, called *harmonic distance*. The stronger the audible interaction between the proximate partials, the closer the relationship between fundamentals will seem. The more equal the proximate partials' volumes, the closer they are to their respective fundamentals, the more likely that a *consonant* intonation may be possible by minimising or eliminating beating.

Harmonic distance may be quantified in various ways. I think it is most usefully modelled by James Tenney's generalised *harmonic space* lattice, with dimensions generated by each prime number. Distances are measured by following the pathways along various axes connecting individual pitches in a network. Compactness and proximity of pitches represented in this harmonic space provide one way to imagine music composed in a *microtonally extended Just Intonation*, which is the practice of conceptualising intervals between tonal sounds as *frequency ratios*, tuning these intervals accordingly, and thereby searching for ways of hearing and composing the manifold tonal relationships between sounds.

Every timbre suggests harmonic constellations, but timbre and harmony ought not to be mistaken for each other; harmony is the *more general* musical principle. Timbre describes the composition, morphology and colour of a specific sound; harmony describes a world of potentially perceivable relationships within the interactions of any combination of tonal sounds. By changing or inflecting timbres, intensities, registers, etc. in composition and in the playing of music, different aspects of the harmonic relationship unfolding in time are highlighted or revealed.

1. Harmonic space, the Pythagorean diatonic and a well-tempered JI tuning

Harmonic space may be extended infinitely and symmetrically from any given pitch. Its exploration is made possible by establishing a notational frame of reference, which enables various proximate relationships of different dimensions to be compared, composed, and played.

Each dimension of harmonic space is generated by the ratio of a prime number to unity (1:p or p:1). Every other ratio may be written as a product of such elementary proportions, which may be interpreted as a collection of steps in harmonic space. The simplest harmonic step is the octave 1:2. An equal division of this ratio into many small intervals produces various *atonal approximations* of the infinite space of all tonal, rational pitch relationships. These irrational constructions are useful as measurement schemes, revealing enharmonic proximities and assisting musicians to come close to hearing and playing tonal pitch relationships *that may actually be heard*.

To approach any rational relationship within the margin of error of an acoustic instrument, the division of the octave into 1200 cents, with the frequency chosen for the tuning reference note A set to 0 cents, is the most practical and compatible with current music practice. On the other hand, to interpret arbitrary cents values harmonically as sequences of steps in harmonic space, it is necessary to make a *reverse mapping*. For any given pitch, what are the simplest rational relationships occurring within a small melodic distance? In other words: what simple combinations of steps in harmonic space return close to their starting point without backtracking?

The notation of harmonic space requires a representation of ratios from an arbitrary origin (1/1). For practical purposes, we may call this origin D, which has the advantage of being symmetrically positioned in the existing system of pitch notation, and on the standard keyboard layout. The simplest harmonic interval that generates *different* pitch-classes is the ratio 1:3 (octave plus fifth). Extended above and below D, and combined with octave transpositions, a Pythagorean series is obtained.

The first seven pitches of this series, symmetric around D, are F-8c C-6c G-4c D-2c A-0c E+2c B+4c. This sequence of pitches, ordered melodically, produces a

Pythagorean diatonic division of the octave into five *major wholetones* — C-D-E and F-G-A-B (all in the proportion 8:9 or 204c) — and two semitone *limmas* — E-F and B-C (both in the proportion 243:256 or 90c). Note that the two limmas, taken together, are approximately 180c or 24c (one *Pythagorean comma*) *less* than a wholetone. Similarly, six Pythagorean wholetones, each consisting of two fifths less one octave, when combined, exceed an octave by 24c.

This Pythagorean diatonic division is compatible with many currently used musical instruments: it matches the natural tuning of the open strings of violins, violas, cellos and basses, viols, guitars and lutes as well as traditional stringing and fretting systems of classical Turkish, Arabic and Persian instruments. So it is the most logical starting point for a common space of tonal sounds.

How may the Pythagorean diatonic be extended to map a more general harmonic space? Continuing the sequence of fifths until there are 13 pitches produces the small Pythagorean comma mentioned above. Therefore, a more complete tonal space with perfect 2:3 fifths will require, as a minimum condition, *a microtonal division into steps approximating one comma*.

Returning to a simpler approach, taking the diatonic pitches once again as starting point, it would also be possible to divide each of the wholetones in two approximately equal parts, obtaining a sequence of 12 semitones and accepting some more or less “out-of-tune” approximate fifths. This is the process that has produced the common 12-note keyboard layout used today. If the seven diatonic notes are established by the Pythagorean diatonic, producing a sequence of six perfect fifths, the remaining five pitches might usefully be chosen so that each additional “fifth” produced is slightly too small by about 1/6 Pythagorean comma, resulting in a well-tempered “circle” of fifths. Another way of looking at this would be to say that each successive semitone in the sequence of fifths — namely above E, A, D, G and below C, G, D, A — must slightly increase in size, ranging from a limma of 90c to a half-tone of 102c equally dividing the 204c G:A interval.

The simplest rational semitones of this size may be obtained from the harmonic series by considering the ratios 16:17 (105c), 17:18 (99c) and 18:19 (94c). To obtain a near-equal rational division of the wholetone G:A (8:9 or 16:18), consider the simplest *epimoric* (n:n+1) division 16:17:18 (G:Ab:A) and also consider its inversion, produced by taking first 17:18 (G:G#) followed by 16:17 (G#:A). The first division is obtained by playing the natural harmonics 16, 17, and 18. The second division is easily produced on a monochord or fretted instrument by dividing the octave in 18 equidistant units. The proportion G:G#:Ab:A, combining both divisions, may be written as (16*17):(16*18):(17*17):(17*18) or 272:288:289:306. By taking the arithmetic mean of 288 and 289, and doubling the numbers, the mean proportion 544:577:612 is obtained. 577 represents the mean semitone which may be interpreted as either G# or Ab. G:G#=Ab with proportion 544:577 is 101.96c and G#=Ab:A with proportion 577:612 is 101.95c, obtaining the two desired half-tones.

Thus, by tuning A:Bb and F#:G as 18:19, D:D# and Db:C as 17:18, and dividing G:A as above, it is possible to complete a well-tempered circle of “fifths” in which all of the notes are harmonically related in a space defined by the mostly-smaller primes 2, 3, 17, 19 and 577 (!). The resulting tuning is excellent in all keys and offers precise and fascinating just harmonies. Of course, the most simple major triads: F, C, and G, are Pythagorean, since there are no 5-limit consonances, but this inversion of historical practice is in fact remarkable and refreshing to my ears. I highly recommend the adoption of this *Just Intonation (JI) well temperament* for solo and chamber music. The beating differences between the variously complex, rationally tuned thirds highlight, to my ears, the falsity at the heart of conventional equal temperament, which disguises and softens its dissonance by imposing a symmetrical uniformity.

The first of my three tables for Bob is a notation of this piano tuning, with a complementary extension into the “quartertone” realm based on harmonics 5, 7, 11 and 13 for an optional second piano. The inspiration for this tuning was the traditional Vallotti well temperament with six Pythagorean fifths and six tempered fifths reduced by 1/6 Pythagorean comma. The more colourful but still well-tempered JI tuning of the second piano, with one fifth even tuned slightly larger than 2:3, reflects the fantasy of early historical systems like the French *tempérament ordinaire*.

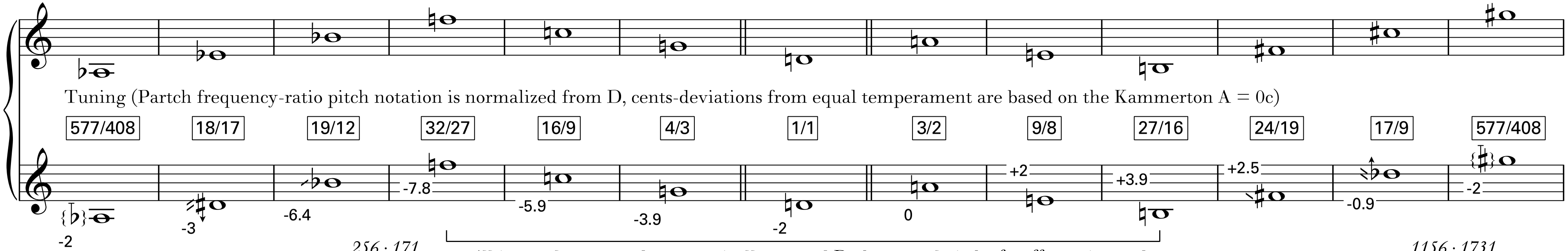
Taken together, both pianos present a spectrum of just “near-thirds” and “near-fifths” producing intervals that echo the various comma-fractional alterations of historical temperaments, but also offering pitches that may be integrated without compromise in a larger mosaic of microtonally extended Just Intonation.

Table 1:
“Well-Tempered” Extended JI Quartertone Tuning for Keyboard Instruments

“Well-Tempered” Extended JI Quartertone Tuning for Keyboard Instruments

based on a Harmonic Space subset defined by the prime partials 3, 5, 7, 11, 13, 17, 19, and 577

Key (tuning repeats in all octaves; either ignore inharmonicities or adjust slightly to reduce beats in the 2 : 3 ratios by tuning unisons between 2nd and 3rd partials)



*577 : 864 = 699c = 2 : 3 * (577 : 576) ca. -1/8 SC*

*216 : 323 = 696.6c = 2 : 3 * (324 : 323) ca. -1/4 SC*

*256 : 171 = 698.6c = 2 : 3 * (513 : 512) ca. -1/6 SC*

All intervals are tuned symmetrically around D, the central pitch of staff notation and traditional keyboard design. The six fifths between white keys are tuned Pythagorean in frequency proportion 2 : 3 = 702c. The six remaining fifths are each made slightly smaller, by approximately 1/6, 1/4 and 1/8 of a Syntonic Comma, producing a “circle” of fifths. The “chromatic” keys divide the Pythagorean wholetones into semitones 16 : 17 (C - Db and E - D#), 17 : 18 (Db - D and D# - D), 18 : 19 (A - Bb and G - F#) and 76 : 81 (Bb - B and F# - F).

*256 : 171 = 698.6c = 2 : 3 * (513 : 512) ca. -1/6 SC*

*216 : 323 = 696.6c = 2 : 3 * (324 : 323) ca. -1/4 SC*

*1156 : 1731 = 699c = 2 : 3 * (578 : 577) ca. -1/8 SC*

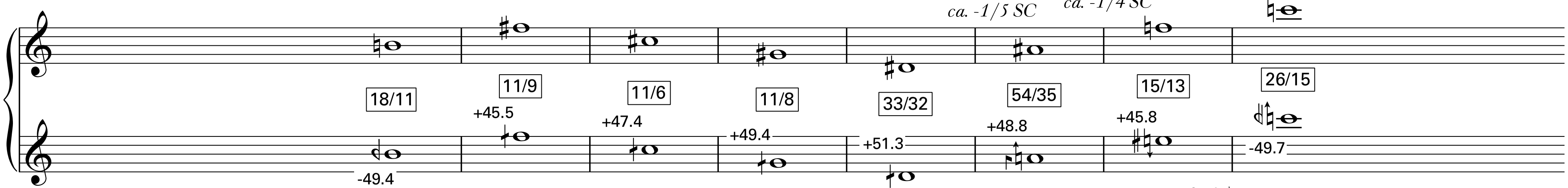
The ratio for G#/Ab is determined by dividing the wholetone G : A arithmetically in the frequency proportion 16 : 17 : 18 (obtaining an Ab) and harmonically in the frequency proportion 272 (=17·16) : 288 (=18·16) : 306 (=18·17) (obtaining a G#). The ratio G# : Ab, 288 : 289 (=17·17) is divided in the proportion 576 : 577 : 578. The intermediate pitch 577 produces a diminished fifth D - Ab of 600.003c, and a tritone Ab - D of 599.997c.

A second keyboard tuned as below augments the well-tempered tuning with quartertones.

*576 : 385 = 697.5c = 2 : 3 * (385 : 384) ca. -1/5 SC*

*175 : 117 = 697c = 2 : 3 * (351 : 350) ca. -1/4 SC*

*338 : 225 = 704.5c = 2 : 3 * (675 : 676) ca. +1/8 SC*

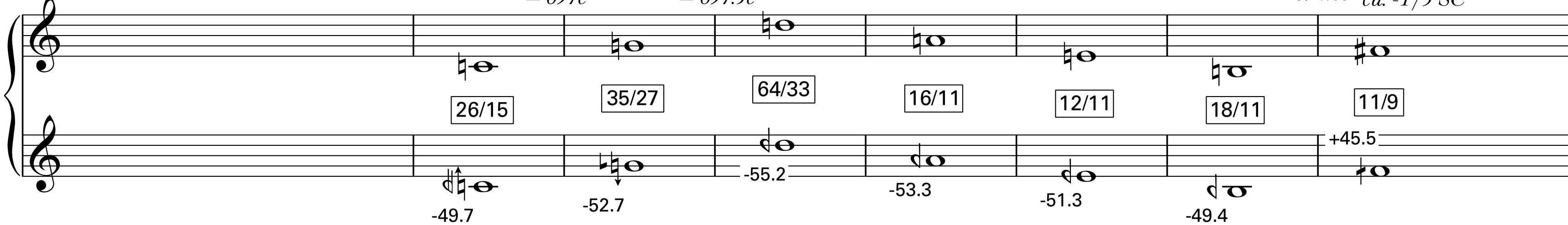


175 : 117 = 697c

576 : 385 = 697.5c

121 : 81 = 694.8c

*= 2 : 3 * (243 : 242) ca. -1/3 SC*



Berlin, 7 December 2015 / 14 February 2016

2. The Euler lattice and primary 23-limit enharmonic proximities

How far is it useful to extend a representation of harmonic space beyond the Pythagorean division? *The Extended Helmholtz-Ellis JI Pitch Notation*, described elsewhere (plainsound.org), allows notation of the harmonic series from any fundamental as far as the 64th partial and beyond. In my experience, some intervals up to at least the 23rd partial may be directly tuned in an appropriately composed register, timbre and musical context. In my recent music, I generally have not exceeded the 23rd partial, and for the most part am exploring intervals and aggregates up to the 13th harmonic. Higher primes above 23 are perhaps more usefully notated by means of cents and a text indication rather than by using special accidentals, which I now prefer to reserve for the more easily perceived lower primes. Therefore, in defining a generally useful subset or region of harmonic space, I would like to provide for a precise notation of intervals that may be tuned directly by ear in a musical context.

At the same time, some pitches or ratios that may easily be heard and tuned in some circumstances end up requiring a visual notation that is excessively laden with signs. This happens because the musical point of reference is not always perceived as monophonic, in the sense of Harry Partch (a fabric derived from and always respecting one originating pitch), but rather as a shifting sequence of reference points, each related to one another. The simplest way to cope with this notational difficulty is to be able to move flexibly between a fixed *absolute* pitch notation and a floating *relative* pitch notation — after all, ancient Greek music had both systems concurrently. The smallest residual commas in a modulation, which then affect a range of subsequent pitches, might be more effectively subsumed into a slight shifting of the *Kammerton*, like a shifting of key. Another approach would be to substitute very near almost-equivalent pitches with a simpler notation, and to indicate the desired harmonic relationships as ratios in the score. In both situations, including exact desired cents values may provide the desired intonation without compromise.

The most consonant basis for harmonic space, and the simplest notational extension of the Pythagorean series, as mentioned above, is a comma-based division. The 5-limit Euler lattice based on primes 2, 3 and 5 produces the network of pitches that underlies both the Indian raga system of *sruti* regions and the European practice of consonant counterpoint. In my 2011 string quartet *Euler Lattice Spirals Scenery* the section *Harmonium for Ben Johnston* maps a complete journey through a central region of 53 pitches combined with their enharmonic peripheral neighbours: 46 additional pitches completing just major and minor triads. I have composed a pathway thorough this space of 99 pitches by linking them in a chain of common-tone triadic modulations with one very small enharmonic transition of 4.2c (10460353203:10485760000).

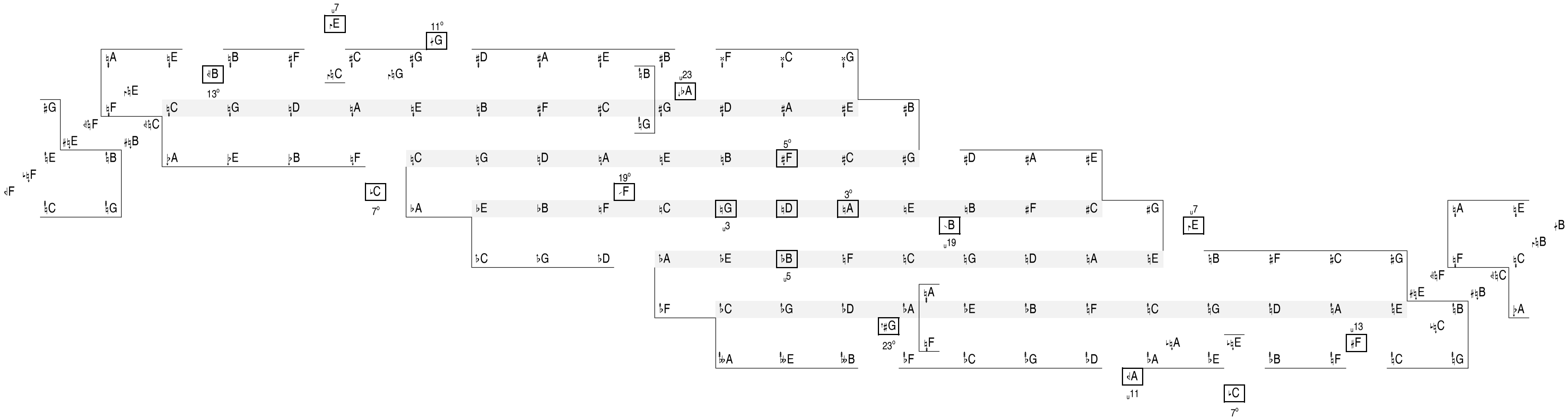
In my second table for Bob, this Euler lattice subset is superimposed with 23-limit harmonic and subharmonic intervals calculated from D, to reveal some of the primary enharmonic proximities in the extended JI universe. The higher prime consonances may be considered as ways of more directly leaping to distant regions of the Euler lattice with slight enharmonic variations of intonation. These relationships are indicated in a notational shorthand for the relationship to a fundamental: D:7° or D^{7o} or D^{7} may be used to indicate an *otonal* relationship, and $_{u}7$:D or $_{u7}D$ or $_{7}D$ may be used to indicate a *utonal* one.

Table 2a:
53-tone 5-limit Euler lattice (shaded) with 46 enharmonic border tones and 23-limit harmonics / subharmonics from D

Table 2b: Some 23-limit enharmonic proximities

53-TONE 5-LIMIT EULER LATTICE (SHADED) WITH 46 ENHARMONIC BORDER TONES & 23-LIMIT HARMONICS / SUBHARMONICS FROM D

Marc Sabat (2016)



Some 23-limit enharmonic proximities

primes			**ratio**	**cents**
3 5	A	G	32768:32805	2.0c
3 5	E	F	10460353203:10485760000	4.2c
3 5	A	A	1594323:1600000	6.2c
3 5 7	E	E	4374:4375	0.4c
3 5 7	F	E	14336000:14348907	1.6c
3 5 7	C	C	5103:5120	5.8c
3 5 7	C	B	224:225	7.7c
3 5 11	G	G	8000:8019	4.1c
3 5 7 11	G	G	384:385	4.5c
3 5 13	B	B	255879:256000	0.8c
3 5 13	B	C	675:676	2.6c
3 5 13	C	B	624:625	2.8c
3 5 13	B	B	324:325	5.3c
3 5 7 13	B	B	4095:4096	0.4c
3 7 13	E	D	728:729	2.4c
3 11 13	G	G	351:352	4.9c
3 11 17	B	A	1088:1089	1.6c
3 19	F	F	512:513	3.4c
3 5 23	G	A	575:576	3.0c

3. A map of 23-limit harmonic space

Investigating very small pitch variations produced by following different harmonic pathways inspired the third table, which is a rising map of 23-limit harmonic space reduced to the simplest combinations of *The Extended Helmholtz-Ellis JI Pitch Notation* (HE accidentals). The idea for this table evolved from my work on the micromelodeon, an algorithm created in 2008 for real-time harmonic microtuning based on projecting tuneable intervals two steps removed from the Pythagorean pentatonic.

In writing music for ensembles of musicians, I am seeking to open awareness of the subtle harmonic variations made possible by considering the intervals of a microtonally extended just intonation. To make the tuning and the harmonic motivation of such intervals clear, I would like to invite musicians to understand both a cents notation and a partial based accidental notation. Even though some pathways composed in my music lead to sequences of more than two accidentals, I find such combinations generally too difficult to read in real-time music making, especially for larger ensembles. So I decided to attempt a reduction of the infinite nesting of rational relationships by limiting the number of microtonal signs.

The comma-based division established by the Euler lattice divides the octave unequally in 53 comma-sized regions. Each diatonic wholetone has approximately nine comma-sized regions, each *apotome* (sharp or flat) five comma-sized regions and each limma four comma-sized regions. Notes altered by three Syntonic commas are only needed as enharmonic completions of major or minor triads, so for the most part, two-comma alterations are sufficient, and the use of double-sharps and -flats is minimised.

Another way of constructing the Euler lattice, arguably a more compact one, would be to have rows of nine pitches, each spanning 8 perfect fifths. For example, the central row could be restricted to the central 9 fifths from Bb to F#. In the row above, the rightmost pitch would be A# lowered by a syntonic comma, lying one schisma (2c) higher than the leftmost Bb in the central row. Similarly, in the row below, the leftmost pitch would be Gb raised by a syntonic comma, one schisma lower than the rightmost F# of the central row. Expanding to a block of seven rows would obtain 63 pitches. Six pairs of pitches are a schisma apart, and four pairs of pitches are in the proportion 15552:15625 or 8.1c apart.

In addition to facilitating readability (less arrows, double-sharps and double-flats), there are several other reasons why I have chosen a different expansion of the 53-comma octave, favouring longer sequences of fifths in five rows set off from each other. My central row runs from Eb to C#, with enharmonic neighbours Ab and G# delineating the Pythagorean comma. The main 53 include, however, the schisma altered versions of these tones: G#-comma-down and Ab-comma-up, producing the most natural transition from Pythagorean to Syntonic intervals when building the *sruti* regions around D. Limits of the single-comma rows are determined by completing the otonal triad on A-comma-up and the utonal trial under G-comma-down. The double-comma rows begin, therefore, from E#-double-down and Cb-double-up, respectively. These are then extended to reach the elegant enharmonic seam dividing the limmas E–F and B–C in four commas and a small remainder of 4.2c (see the second interval noted in Table 2b and mentioned above).

To make the following map of harmonic space written in a simplified subset of HE accidentals I decided to limit the combinations to two signs, in a manner I expect any musician might readily learn to read intuitively. Many complex combinations may be closely approximated, within a few cents, by simpler ones. Small corrections could nevertheless produce any desired interval, which may be notated in the music as a ratio. Other pitches are perhaps most easily notated by either subsuming accidentals into a *Kammerton* shift, as noted above, or by means of ratio and cents indications. The large boxed numbers indicate 53 regions of approximately one comma, dividing the Pythagorean diatonic in five nine-comma wholetones and two four-comma limmas. The Euler lattice is combined with an otonal and utonal expansion from D, including the most clearly perceptible tuneable intervals and epimoric ratios up to the Syntonic comma.

This third table is a just intonation counterpart to the approximation of pitches in cents: a reverse-lookup table mapping cents to frequency ratios. It suggests one model of how combinations of the simplest prime partials, taking only a few harmonic steps from the central Pythagorean region of harmonic space, establish manifold enharmonic proximities and create a flowing complex design of unfolding harmonic relationships.

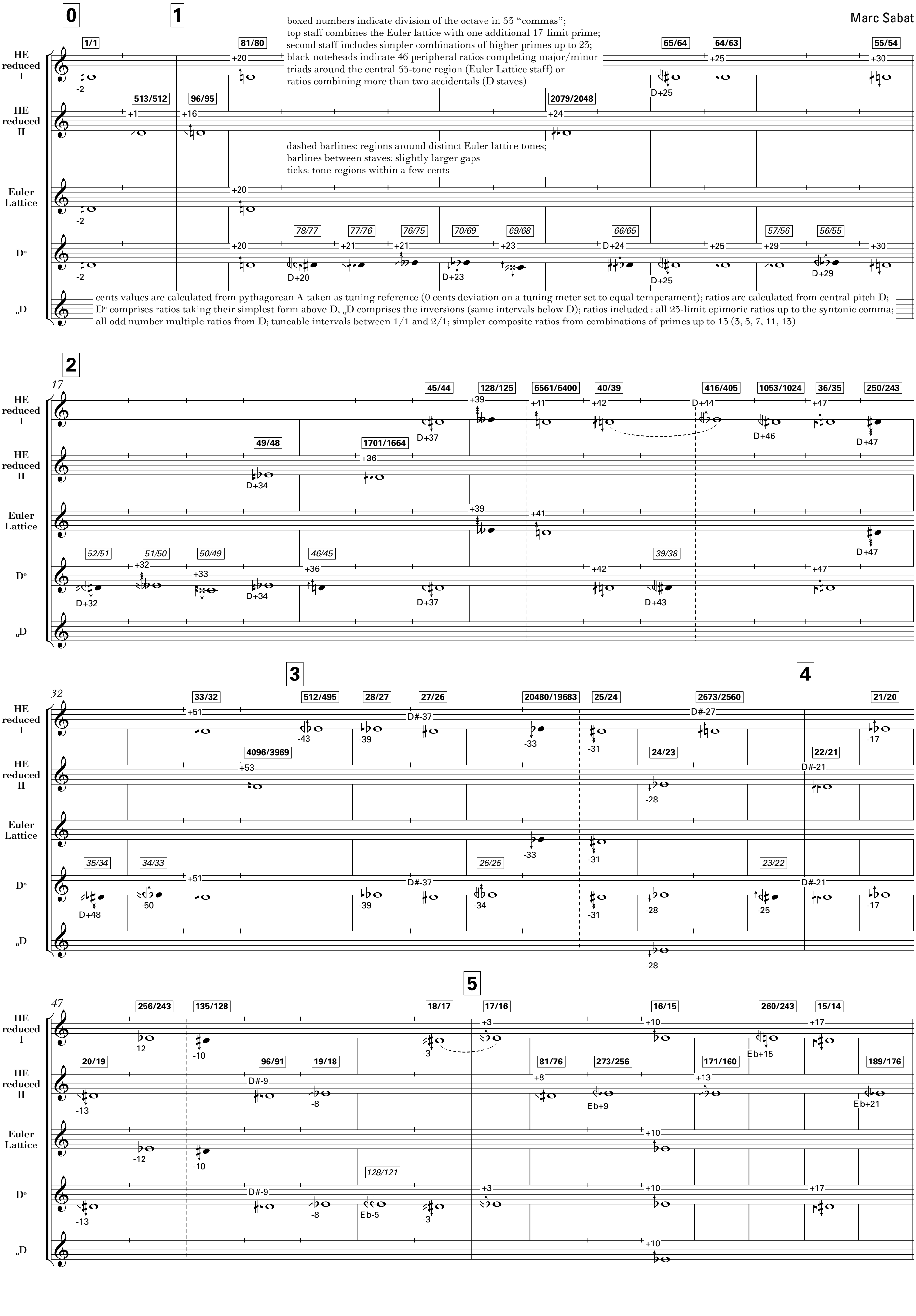
Table 3:
Euler Lattice with sruti regions from D represented by the simplest combinations of Helmholtz-Ellis accidentals

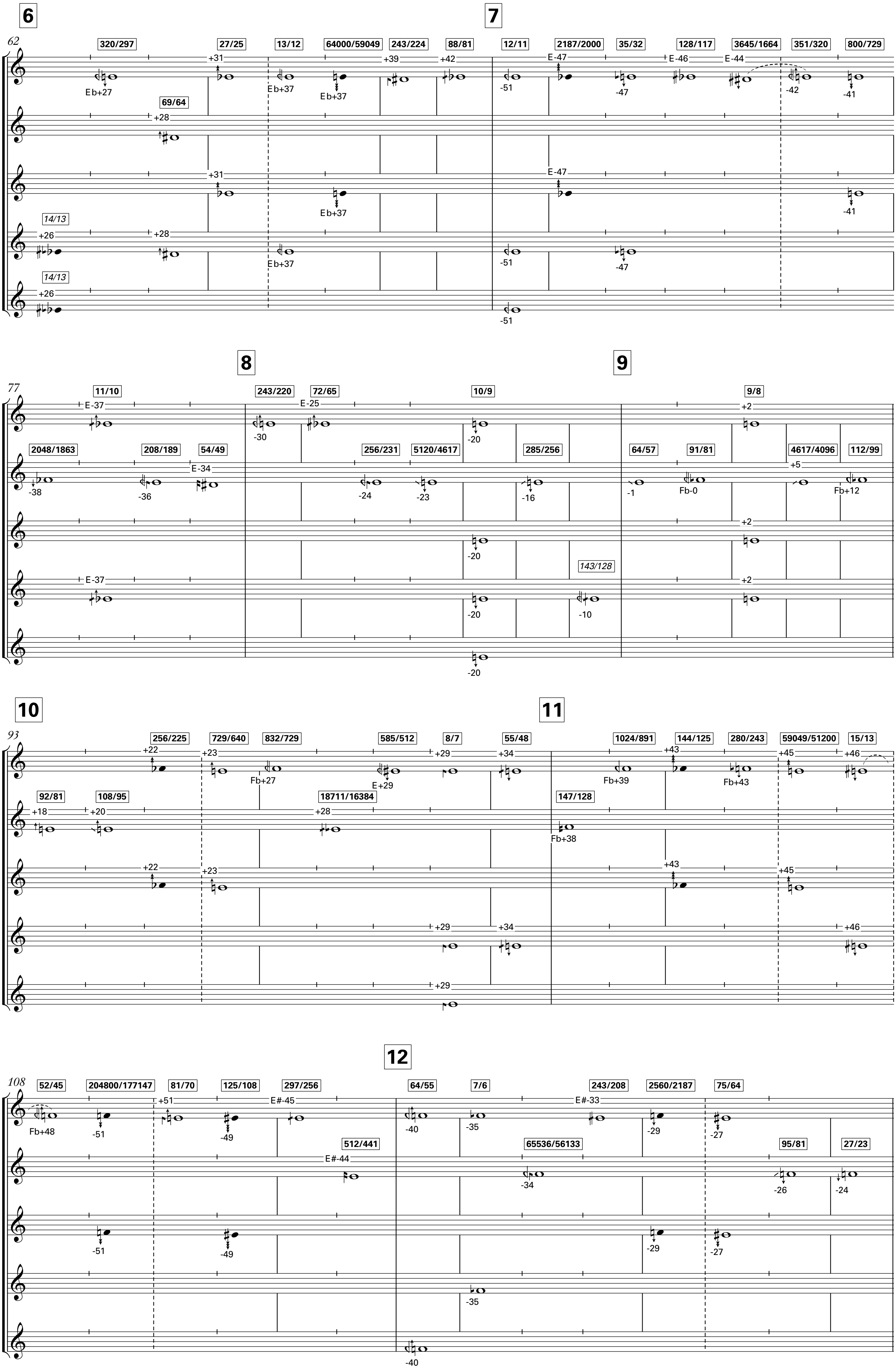
Euler lattice with sruti regions from D represented by the simplest combinations of Helmholtz-Ellis accidentals

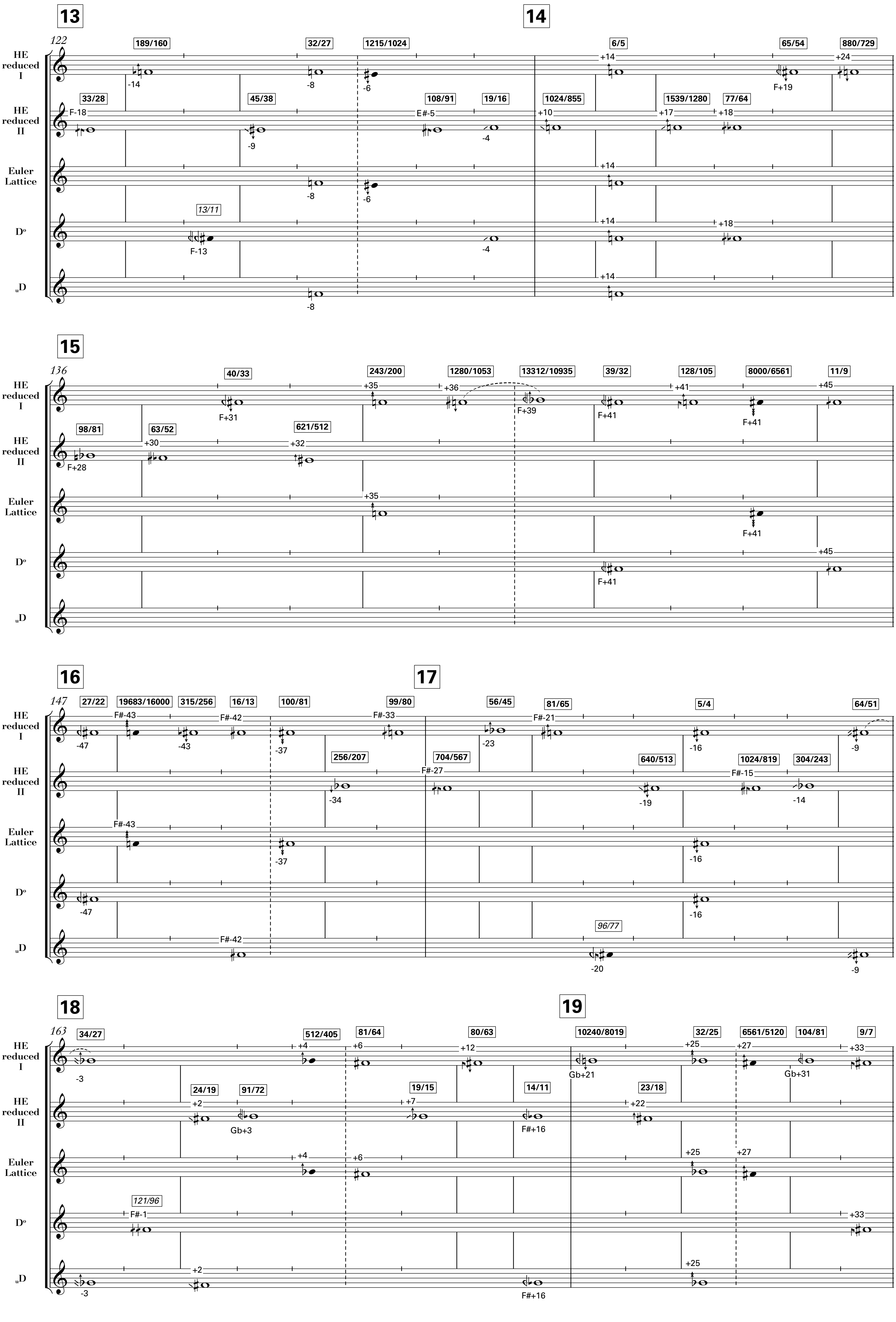
a map of 23-limit harmonic space

for Bob Gilmore

Marc Sabat




6
7
62
HE reduced I
HE reduced II
Euler Lattice
Dº
ᵤD
320/297
27/25
13/12
64000/59049
243/224
88/81
12/11
2187/2000
35/32
128/117
3645/1664
351/320
800/729
Eb+27
+31
Eb+37
Eb+37
+39
+42
-51
E-47
-47
E-46
E-44
-42
-41
69/64
+28
+31
Eb+37
E-47
-41
14/13
+26
+28
Eb+37
-51
-47
14/13
+26
-51
8
9
77
11/10
243/220
72/65
10/9
9/8
E-37
-30
E-25
-20
+2
2048/1863
208/189
54/49
256/231
5120/4617
285/256
64/57
91/81
4617/4096
112/99
-38
-36
E-34
-24
-23
-16
-1
Fb-0
+5
Fb+12
+2
-20
143/128
E-37
-20
-10
+2
-20
10
11
93
256/225
729/640
832/729
585/512
8/7
55/48
1024/891
144/125
280/243
59049/51200
15/13
+22
+23
Fb+27
E+29
+29
+34
Fb+39
+43
Fb+43
+45
+46
92/81
108/95
18711/16384
147/128
+18
+20
+28
Fb+38
+22
+23
+43
+45
+29
+34
+46
+29
12
108
52/45
204800/177147
81/70
125/108
297/256
64/55
7/6
243/208
2560/2187
75/64
Fb+48
-51
+51
-49
E#-45
-40
-35
E#-33
-29
-27
512/441
65536/56133
95/81
27/23
E#-44
-34
-26
-24
-51
-49
-29
-27
-35
-40



13
14
122
HE reduced I
HE reduced II
Euler Lattice
Dº
ᵤD
189/160
32/27
1215/1024
6/5
65/54
880/729
-14
-8
-6
+14
F+19
+24
33/28
45/38
108/91
19/16
1024/855
1539/1280
77/64
F-18
E#-5
+10
+17
+18
-9
-4
13/11
F-13
15
136
40/33
243/200
1280/1053
13312/10935
39/32
128/105
8000/6561
11/9
+35
+36
F+31
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F+41
+41
+45
98/81
63/52
621/512
+30
+32
F+28
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315/256
16/13
100/81
99/80
56/45
81/65
5/4
64/51
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-34
-19
-14
96/77
-20
18
19
163
34/27
512/405
81/64
80/63
10240/8019
32/25
6561/5120
104/81
9/7
+4
+6
+12
+25
+27
+33
-3
Gb+21
Gb+31
24/19
91/72
19/15
14/11
23/18
+2
+7
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Gb+3
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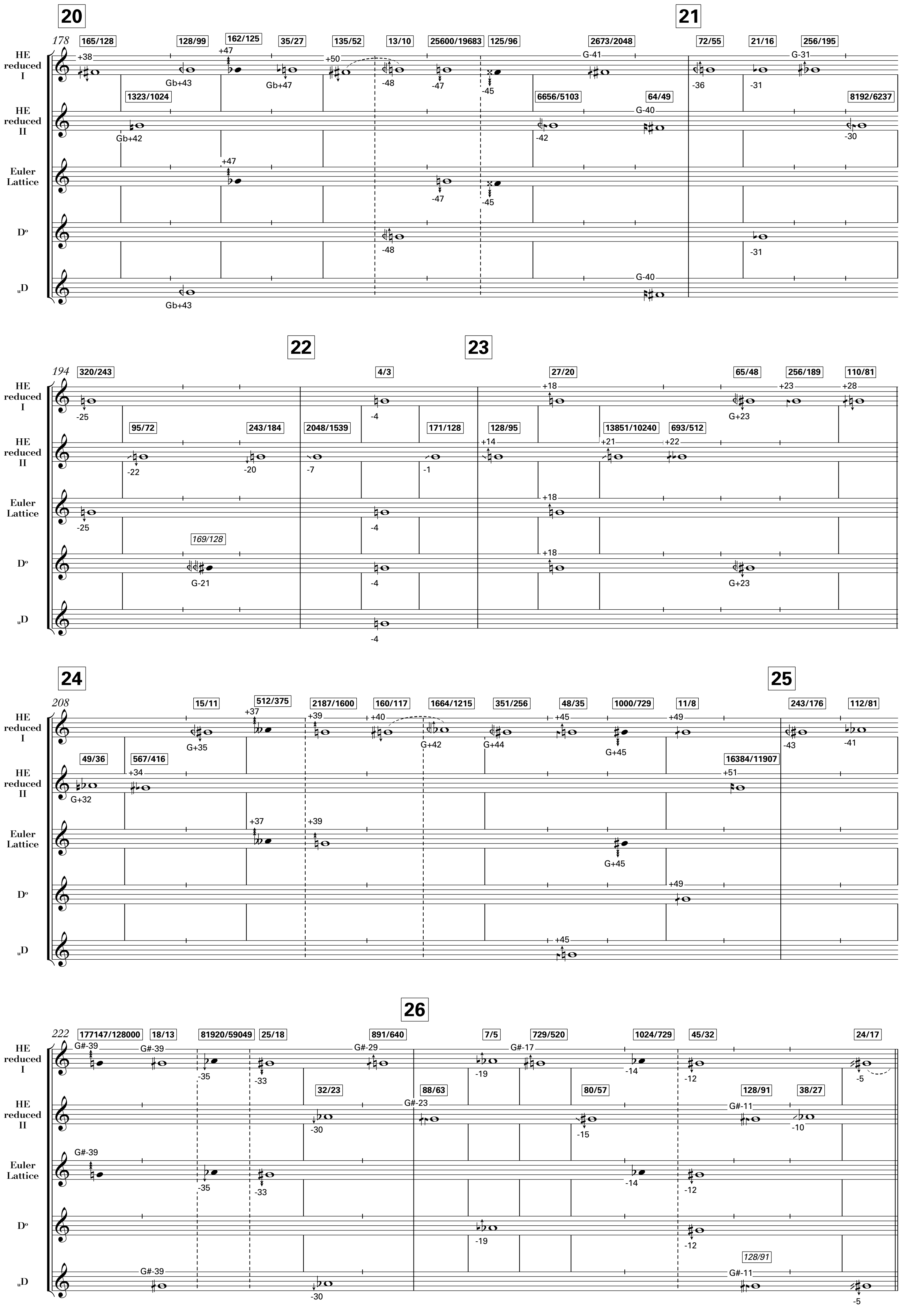
F#-1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21/16
256/195
1323/1024
6656/5103
64/49
8192/6237
HE reduced I
HE reduced II
Euler Lattice
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23
194
320/243
4/3
27/20
65/48
256/189
110/81
95/72
243/184
2048/1539
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128/95
13851/10240
693/512
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512/375
2187/1600
160/117
1664/1215
351/256
48/35
1000/729
11/8
243/176
112/81
49/36
567/416
16384/11907
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222
177147/128000
18/13
81920/59049
25/18
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7/5
729/520
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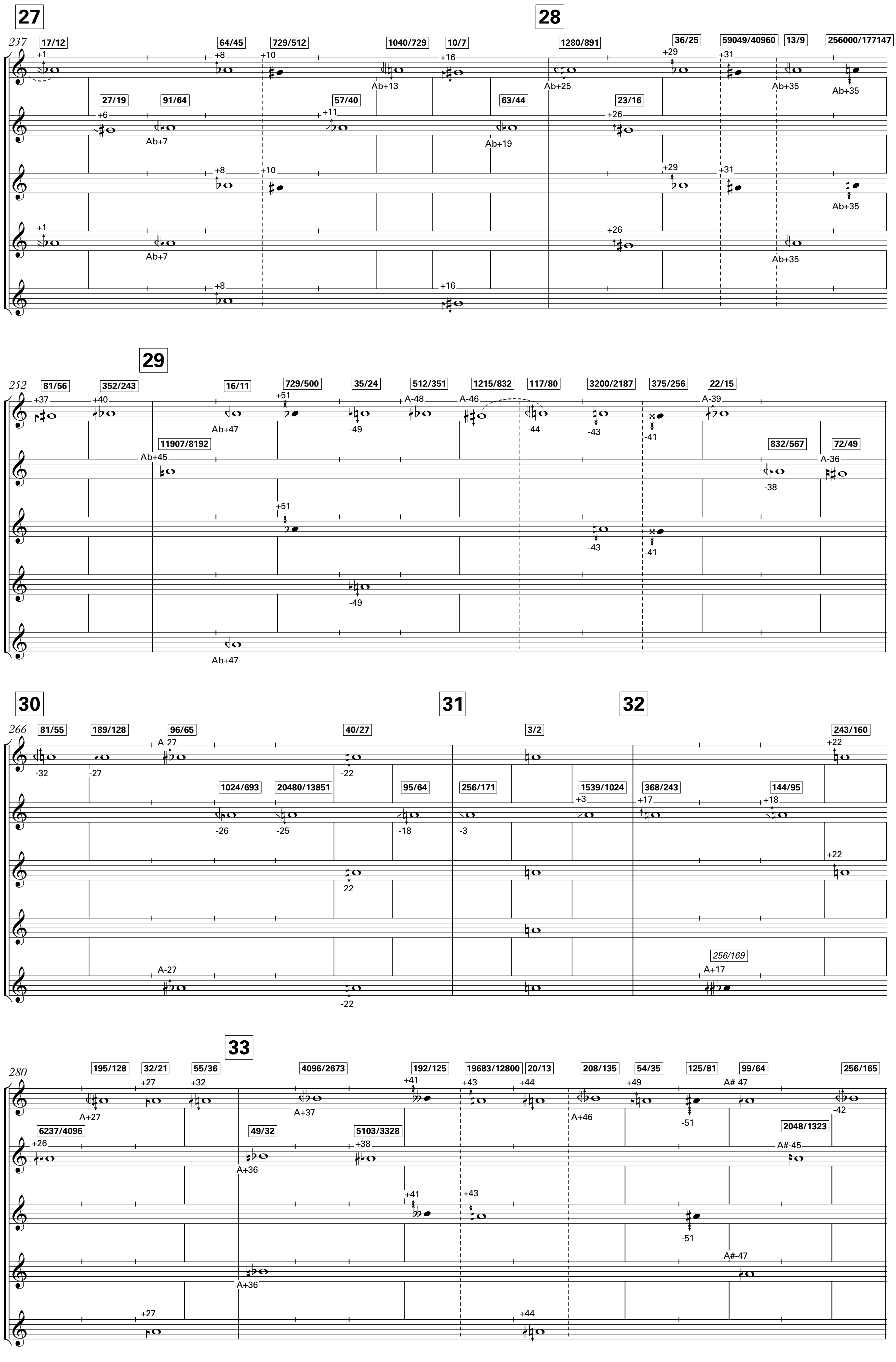
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27/19
91/64
57/40
63/44
23/16
HE reduced I
HE reduced II
Euler Lattice
Dº
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252
81/56
352/243
16/11
729/500
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512/351
1215/832
117/80
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81/55
189/128
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40/27
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4096/2673
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2048/1323


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296
HE reduced I
HE reduced II
Euler Lattice
Dº
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14/9
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19/12
192/121
A#-35
A#-25
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13/8
512/315
32000/19683
44/27
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819/512
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207/128
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Bb+39
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338
729/440
108/65
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32/19
91/54
76/45
56/33
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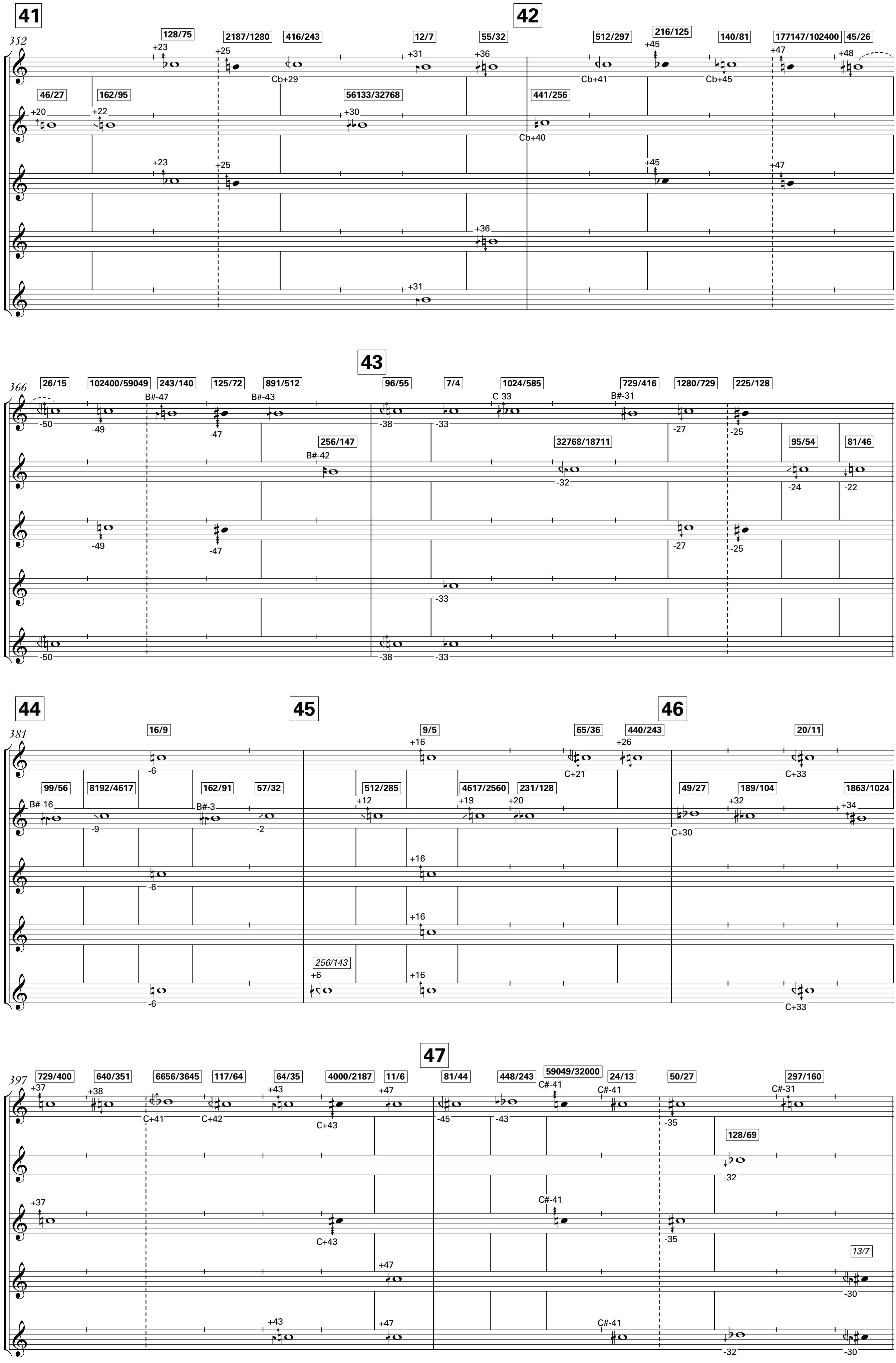
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HE reduced I
HE reduced II
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416/243
12/7
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512/297
216/125
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56133/32768
441/256
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366
26/15
102400/59049
243/140
125/72
891/512
96/55
7/4
1024/585
729/416
1280/729
225/128
256/147
32768/18711
95/54
81/46
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16/9
9/5
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440/243
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99/56
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729/400
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Berlin, 30 March 2016